On Some Explicit Formulas for Bernoulli Numbers and Polynomials

Lazhar Fekih-Ahmed

École Nationale d'Ingénieurs de Tunis, BP 37, Le Belvédère 1002 , Tunis, Tunisia

Abstract

We provide direct elementary proofs of several explicit expressions for Bernoulli numbers and Bernoulli polynomials. As a byproduct of our method of proof, we provide natural definitions for generalized Bernoulli numbers and polynomials of complex order.

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1. Introduction

As Gould pointed out in his survey article [5], a well-known explicit formula for Bernoulli numbers, which dates back to Worpitzky [10], is given by the double sum

$$B_k = \sum_{n=0}^{k} \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k, k \geq 0, \tag{1}$$

where $\binom{n}{j}$ is the binomial coefficient. The first few values of formula (1) are $B_0 = 1$ $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$, $B_8 = -1/30$ etc.

For the sake of convenience and to agree with our notation, the lower limits of summation in both sums in (1) will be changed so that the above sum is given by the following equivalent form

Email address: lazhar.fekihahmed@enit.rnu.tn (Lazhar Fekih-Ahmed)

$$B_k = \sum_{n=1}^{k} \frac{1}{n+1} \sum_{j=1}^{n} (-1)^j \binom{n}{j} j^k, k \geq 1. \tag{2}$$

If we define the forward differences $\Delta_n(k)$ by

$$\Delta_n(k) = \sum_{j=1}^{n} (-1)^j \binom{n}{j} j^k, \tag{3}$$

equation (2) can be rewritten as

$$B_k = \sum_{n=1}^{k} \frac{1}{n+1} \Delta_n(k), k \geq 1. \tag{4}$$

In this note, our main result is proving the following two explicit formulas for Bernoulli numbers

$$B_k = (-1)^{k+1} \sum_{n=1}^{k} \frac{1}{n(n+1)} \Delta_n(k), k \geq 1, \tag{5}$$

and

$$B_k = (-1)^{k+1} \sum_{n=1}^{k+1} \frac{1}{n^2} \Delta_n(k+1), k \geq 0. \tag{6}$$

While formula (1) is a well-known explicit formula and several proofs have been provided by different authors, the two formulas (5) and (6) are less-known[1].

As a byproduct of our method of proof we provide extensions to Bernoulli polynomials. We further provide natural definitions for generalized Bernoulli numbers and polynomials of complex order.

[1] Formula (5) has been given in [10, formula (37)]. The same formula is mentioned in [8, formula LXV on page 83]. Formula (6) is also mentioned in [8, formula LXIII on page 82]. The proofs in [8] use the identity $\Delta_n(k) = n(\Delta_n(k-1) - \Delta_{n-1}(k-1))$ and the property that odd-indexed Bernoulli numbers have zero values. In [8, page 83] the identity is written in terms of a_n^k as $a_n^k = n(a_n^{k-1} + a_{n-1}^{k-1})$ so that $\Delta_n(k) = (-1)^n a_n^k = (-1)^n n! S(k,n)$, where $S(k,n)$ are Stirling numbers of the second kind.

2. Fractional Derivatives

Suppose that the function $\psi(t)$ is holomorphic and that $\lim_{t\to-\infty}\psi(t) = 0$. According to Laurent [7], the fractional derivative of order $\alpha \in \mathbb{C}$ between the points $-\infty$ and $x \in \mathbb{R}$ of the function $\psi(t)$ is given by the contour integral[2]

$$I^\alpha \psi(x) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{\mathcal{C}} \frac{\psi(t)}{(t-x)^{\alpha+1}}\, dt, \tag{7}$$

where $\mathcal{C}$ is the Hankel contour consisting of the three parts $C = C_- \cup C_\epsilon \cup C_+$: a path which extends from $(-\infty, -\epsilon)$, around the point x counter clockwise on a circle of center the point x and of radius ϵ and back to $(-\epsilon, -\infty)$, where ϵ is an arbitrarily small positive number.

When $\mathrm{Re}(1+\alpha) \geq 0$, there is no ambiguity in the definition of $I^\alpha\psi(x)$. The integrals along C_- and C_+ cancel each other. $I^\alpha\psi(x)$ is thus equal to the integral along C_ϵ and this integral can be easily evaluated by residue calculus. In particular, when $\alpha = 0$, formula (7) is simply Cauchy's formula:

$$I^0\psi(x) = \psi(x), \tag{8}$$

and when $\alpha = n$ is a positive integer, Laurent's contour integral $I^\alpha\psi(x)$ gives the classical derivative of $\psi(t)$ at the point x:

$$I^n\psi(x) = \psi^{(n)}(x). \tag{9}$$

When $\mathrm{Re}(1+\alpha) < 1$, the portion of the integral along the circle C_ϵ is zero. The integral along the remaining portions of the contour is estimated using a proper determination of the multi-valued function $(t-x)^{-\alpha-1}$. If we choose the cut long the semi-axis $(-\infty, x)$, then $(t-x)^{-\alpha-1} = e^{-(\alpha+1)(\log(t-x)-i\pi)}$ along C_- and $(t-x)^{-\alpha-1} = e^{-(\alpha+1)(\log(t-x)+i\pi)}$ along C_+, where $\log(t-x)$ is purely real when $t-x>0$. Moreover, $t = re^{-i\pi}$ along C_- and $t = re^{i\pi}$ along C_+, as r varies from ϵ to $+\infty$. The integral (7) becomes

$$I^\alpha\psi(x) = \frac{\Gamma(1+\alpha)}{2\pi i}(e^{-\alpha\pi i} - e^{\alpha\pi i}) \int_x^\infty \psi(-r)(r-x)^{-\alpha-1}\, dr. \tag{10}$$

Finally, using the reflection formula of the Gamma function, we obtain

[2]The derivative can of course be defined for all $x \in \mathbb{C}$.

$$I^\alpha \psi(x) = \frac{1}{\Gamma(-\alpha)} \int_x^\infty \psi(-r)(r-x)^{-\alpha-1}\, dr. \tag{11}$$

3. Specializing to $\zeta(s)$

For the particular case $x = 0$ and $\alpha = -s$, the fractional derivative of order $-s$ at zero is given by

$$I^{-s}\psi(0) = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \psi(t) t^{s-1}\, dt \tag{12}$$

when $\mathrm{Re}(1-s) \geq 0$, and by

$$I^{-s}\psi(0) = \frac{1}{\Gamma(s)} \int_0^\infty \psi(-t) t^{s-1}\, dt \tag{13}$$

when $\mathrm{Re}(s) > 0$.

Now by an appropriate choice of $\psi(t)$, we will be able to write the Riemann zeta function as the fractional derivative of $\psi(t)$ at 0. Indeed, let

$$\psi(-t) = \phi(t) = \frac{d}{dt}\left(\frac{-t}{e^t - 1}\right) = \frac{te^{-t}}{(1-e^{-t})^2} - \frac{e^{-t}}{1-e^{-t}}. \tag{14}$$

We have shown in [3] that the Riemann zeta function has the integral representation[3]

$$(s-1)\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \phi(t) t^{s-1}\, dt, \mathrm{Re}(s) > 0, \tag{15}$$

and that for all $s \in \mathbb{C}$

$$(s-1)\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \psi(t) t^{s-1}\, dt, \tag{16}$$

where $\mathcal{C}$ is the same Hankel contour used in equation (7). Comparing with equations (12) and (8), we easily obtain

$$I^{-s}[\psi(t)]_{t=0} = (s-1)\zeta(s). \tag{17}$$

[3]The definition of $\phi(t)$ used here is different from the one in the cited paper.

That is, $(s-1)\zeta(s)$ is simply the fractional derivative of order $-s$ of the function $\psi(t)$ at the origin. Furthermore, for an integer $k \geq 2$, the derivative of order $k-1$ of $\psi(t)$ at the point $t=0$ is, by Laurent's definition, given by

$$I^{k-1}[\psi(t)]_{t=0} = \psi^{(k-1)}(0) = \frac{\Gamma(k)}{2\pi i}\int_{\mathcal{C}} \psi(t) t^{-k}\, dt. \tag{18}$$

Having achieved this, we know also that the Bernoulli numbers are usually defined using the generating function

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k, \quad |t| < 2\pi. \tag{19}$$

These numbers can also be defined in terms of the function $\psi(t)$ instead, since we have

$$\psi(-t) = \frac{d}{dt}\Big(\frac{-t}{e^t-1}\Big) = \sum_{k=1}^{\infty} \frac{-B_k}{(k-1)!} t^{k-1}. \tag{20}$$

Therefore,

$$\begin{aligned} B_k &= (-1)^k \psi^{(k-1)}(0) = (-1)^k I^{k-1}[\psi(t)]_{t=0} = (-1)^{k+1} k\zeta(1-k), \quad \text{or} \\ B_k &= -\phi^{(k-1)}(0) = -I^{k-1}[\phi(-t)]_{t=0} = k\zeta(1-k). \end{aligned} \tag{21}$$

This last equation is the basis of all our subsequent derivations. It relates the Bernoulli numbers, the Riemann-zeta function and fractional derivatives in a single equation.

4. The First Explicit Formula for B_k

In [3], we have also obtained a globally convergent series representation of the Riemann zeta function. It is given by the formula

$$(s-1)\zeta(s) = \sum_{n=1}^{\infty} \frac{S_n(s)}{n+1}, \qquad \text{with} \tag{22}$$

$$S_n(s) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+1)^{-s} \text{ for } n \geq 2, \tag{23}$$

and $S_1(s) = 1$. We have also shown that when $\mathrm{Re}(s) > 0$ the sum $S_n(s)$ can be rewritten as

$$S_n(s) = \frac{1}{\Gamma(s)} \int_0^\infty (1 - e^{-t})^{n-1} e^{-t} t^{s-1}\, dt, \tag{24}$$

and that the function $\phi(t)$ verifies

$$\phi(t) = \sum_{n=1}^{\infty} \frac{(1 - e^{-t})^{n-1} e^{-t}}{n+1} \tag{25}$$

uniformly[4] for $0 < t < \infty$.

Because of equation (24), the definition of fractional derivative (15) and uniform convergence, we may interchange summation and integration inside the integral sign[5]. Thus, we may apply the fractional derivative operator termwise to obtain

$$I^{-s}[\psi(t)]_{t=0} = I^{-s}[\phi(-t)]_{t=0} = \sum_{n=1}^{\infty} \frac{I^{-s}[(1 - e^{t})^{n-1} e^{t}]_{t=0}}{n+1}. \tag{27}$$

Particularly, for $-s = k - 1$ we obtain

$$\begin{aligned} I^{k-1}[\phi(-t)]_{t=0} &= \sum_{n=1}^{\infty} \frac{I^{k-1}[(1 - e^{t})^{n-1} e^{t}]_{t=0}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{d^{k-1}}{dt^{k-1}} [(1 - e^{t})^{n-1} e^{t}]_{t=0} \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \frac{d^{k}}{dt^{k}} [(1 - e^{t})^{n}]_{t=0}. \end{aligned} \tag{28}$$

But $\frac{d^k}{dt^k}[(1 - e^t)^n]_{t=0} = 0$ if $n \geq k + 1$. Therefore, the infinite sum in (28) reduces to a finite sum

[4]When $\mathrm{Re}(s) < 1$, $S_n(s)$ can obviously be written as

$$S_n(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} (1 - e^{t})^{n-1} e^{t} t^{s-1}\, dt. \tag{26}$$

[5]This have been rigourously proved in [3].

$$
\begin{aligned}
I^{k-1}[\phi(-t)]_{t=0} &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \frac{d^k}{dt^k}[(1-e^t)^n]_{t=0} \\
&= \sum_{n=1}^{k} \frac{1}{n(n+1)} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{d^k}{dt^k}[e^{jt}]_{t=0} \\
&= (-1)^k \sum_{n=1}^{k} \frac{1}{n(n+1)} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k \\
&= (-1)^k \sum_{n=1}^{k} \frac{1}{n(n+1)} \Delta_n(k). \qquad (29)
\end{aligned}
$$

The last equation combined with (21) gives the explicit formula for B_k:

$$
B_k = (-1)^{k+1} \sum_{n=1}^{k} \frac{1}{n(n+1)} \Delta_n(k), k \geq 1. \qquad (30)
$$

5. Bernoulli Polynomials $B_k(1-x)$

The generalization of the series formula (22) for the Hurwitz zeta function $\zeta(s,x)$ defined for $0 < x \leq 1$ by

$$
\zeta(s,x) = \sum_{n=1}^{\infty} \frac{1}{(n-1+x)^s} \qquad (31)
$$

is given in [4] by the formula

$$
(s-1)\zeta(s,x) = \sum_{n=1}^{\infty} S_n(s,x) \Big(\frac{1}{n+1} + \frac{x-1}{n} \Big), \qquad (32)
$$

where $S_n(s,x)$ is the generalization of the sums $S_n(s)$:

$$
S_n(s,x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+x)^{-s} \text{ for } n \geq 2. \qquad (33)
$$

There is also a corresponding integral given by

$$
(s-1)\zeta(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \phi_x(t) t^{s-1}\, dt, \qquad (34)
$$

where $\phi_x(t)$ is defined by

$$\begin{aligned}\phi_x(t) &= \frac{te^{-xt}}{(e^t-1)^2} - \frac{e^{-(x-1)t}}{e^t-1} + \frac{(x-1)te^{-(x-1)t}}{e^t-1} \\ &= \frac{d}{dt}\Big(\frac{-t}{e^t-1}\Big)e^{-(x-1)t} + \frac{(x-1)te^{-(x-1)t}}{e^t-1} \\ &= \frac{d}{dt}\Big(\frac{-te^{-(x-1)t}}{e^t-1}\Big). \end{aligned} \tag{35}$$

The formula for Bernoulli polynomials is obtained by repeating exactly the same steps of the previous section. We will repeat these steps for the sake of clarity.

The Bernoulli polynomials are usually defined using the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{nk=0}^{\infty} \frac{B_k(x)}{k!}t^k, \quad |t| < 2\pi, \tag{36}$$

or in terms of the function $\phi_x(t)$ as a generating function

$$\phi_x(t) = \frac{d}{dt}\Big(\frac{-te^{-(x-1)t}}{e^t-1}\Big) = \sum_{k=1}^{\infty} \frac{-B_k(1-x)}{(k-1)!}t^{k-1}. \tag{37}$$

Hence, by using the well-known identity $B_k(1-x) = (-1)^k B_k(x)$, we finally obtain

$$(-1)^k B_k(x) = B_k(1-x) = -\phi_x^{(k-1)}(0) = -I^{k-1}[\phi_x(-t)]_{t=0}. \tag{38}$$

Since, for $0 < t < \infty$,

$$\phi_x(t) = \sum_{n=1}^{\infty} \Big(\frac{1}{n+1} + \frac{x-1}{n}\Big)(1-e^{-t})^{n-1}e^{-xt} \tag{39}$$

uniformly, we may apply the fractional derivative operator termwise to obtain

$$I^{-s}[\phi_x(-t)]_{t=0} = \sum_{n=1}^{\infty} I^{-s}\Big(\frac{1}{n+1} + \frac{x-1}{n}\Big)[(1-e^t)^{n-1}e^{xt}]_{t=0}. \tag{40}$$

For $-s = k-1$,

$$\begin{aligned} I^{k-1}[\phi_x(-t)]_{t=0} &= \sum_{n=1}^{\infty} I^{k-1}\Big(\frac{1}{n+1}+\frac{x-1}{n}\Big)[(1-e^t)^{n-1}e^{xt}]_{t=0} \\ &= \sum_{n=1}^{\infty}\Big(\frac{1}{n+1}+\frac{x-1}{n}\Big)\frac{d^{k-1}}{dt^{k-1}}[(1-e^t)^{n-1}e^{xt}]_{t=0} \quad (41) \end{aligned}$$

But

$$\frac{d^{k-1}}{dt^{k-1}}[(1-e^t)^{n-1}e^{-xt}]_{t=0} = 0, \text{ for n} \geq \text{k}+1 \text{ and}$$
$$(1-e^t)^{n-1}e^{xt} = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{k}e^{(j+x)t};$$

therefore, the infinite sum in (36) reduces to a finite sum

$$\begin{aligned} I^{k-1}[\phi_x(-t)]_{t=0} &= \sum_{n=1}^{k}\Big(\frac{1}{n+1}+\frac{x-1}{n}\Big)\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\frac{d^{k-1}}{dt^{k-1}}[e^{(j+x)t}]_{t=0} \\ &= (-1)^{k-1}\sum_{n=1}^{k}\Big(\frac{1}{n+1}+\frac{x-1}{n}\Big)\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(j+x)^{k-1} \\ &= (-1)^{k-1}\sum_{n=1}^{k}\Big(\frac{1}{n(n+1)}+\frac{x-1}{n^2}\Big)\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}j(j+x-1)^{k-1} \\ &= (-1)^{k}\sum_{n=1}^{k}\Big(\frac{1}{n(n+1)}+\frac{x-1}{n^2}\Big)\Delta_{n,x}(k), \quad (42) \end{aligned}$$

where

$$\Delta_{n,x}(k) = \sum_{j=1}^{n}(-1)^j\binom{n}{j}j(j+x-1)^{k-1}. \quad (43)$$

The last equation combined with (38) gives the explicit formula for $B_k(1-x)$:

$$B_k(1-x) = (-1)^{k+1} \sum_{n=1}^{k} \Big(\frac{1}{n(n+1)} + \frac{x-1}{n^2}\Big) \Delta_{n,x}(k), k \geq 1. \qquad (44)$$

6. Bernoulli Numbers and Polynomials of Complex Index s

There are many generalizations of integer-indexed Bernoulli numbers and polynomials to complex-indexed quantities. The reader may consult for example [1] or [9] and the references therein. Here, we approach the generalization using fractional derivatives.

The Bernoulli numbers and polynomials $B_s(x)$ for s complex can be defined using the fractional derivative operator of order $1-s$ (i.e. replace $k-1$ by $s-1$). When $\mathrm{Re}(s) > 0$ equation (13) applies. Formulas (24) and (25) yield the following natural definition when $\mathrm{Re}(s) > 0$:

$$\begin{aligned} B_s = -I^{s-1}[\phi(-t)]_{t=0} &= -\sum_{n=1}^{\infty} \frac{I^{s-1}[(1-e^t)^{n-1}e^t]_{t=0}}{n+1} \\ &= -\sum_{n=1}^{\infty} \frac{S_n(1-s)}{n+1} \\ &= s\zeta(1-s). \end{aligned} \qquad (45)$$

As for Bernoulli polynomials, their extension is obtained as follows

$$\begin{aligned} B_s(1-x) = -I^{s-1}[\phi_x(-t)]_{t=0} &= -\sum_{n=1}^{\infty} \Big(\frac{1}{n+1} + \frac{x-1}{n}\Big) I^{s-1}[(1-e^t)^{n-1}e^{xt}]_{t=0} \\ &= -\sum_{n=1}^{\infty} \Big(\frac{1}{n+1} + \frac{x-1}{n}\Big) I^{s-1}[\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{k} e^{(j+x)t}]_{t=0} \\ &= -\sum_{n=1}^{\infty} \frac{S_n(1-s,x)}{n+1} \\ &= s\zeta(1-s,x), \end{aligned} \qquad (46)$$

Thus, from equation (46), we see that the Bernoulli polynomials extend naturally to the entire function $s\zeta(1-s,x)$. This is an illustration that entire functions are natural generalization of polynomials.

7. The Second Explicit Formula for B_k

In this section we will prove formula (6) using the globally convergent series representation of the Riemann zeta function discovered in [6]. The series is given by

$$s\zeta(s+1) = \sum_{n=0}^{\infty} \frac{S_{n+1}(s)}{n+1}, \tag{47}$$

$S_n(s)$ being defined in (23).

Ii is easy to show that the sum $S_{n+1}(s)$ can be rewritten as

$$S_{n+1}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} (1-e^{-t})^n e^{-t} t^{s-1}\, dt, \tag{48}$$

and that for $\mathrm{Re}(s) > 0$,

$$s\zeta(s+1) = \frac{1}{\Gamma(s)} \int_0^{\infty} \psi(t) t^{s-1}\, dt, \tag{49}$$

where the function $\psi(t)$ is given by

$$\eta(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{(1-e^{-t})^n e^{-t}}{n+1}. \tag{50}$$

Using the generating function $\psi(t)$, the Bernoulli numbers are now given by

$$B_k = \eta^{(k)}(0) = -I^k[\eta(-t)]_{t=0}. \tag{51}$$

Again, we may apply the fractional derivative operator $(-s = k)$ termwise to obtain

$$\begin{aligned} I^k[\eta(-t)]_{t=0} &= \sum_{n=0}^{\infty} \frac{I^k[(1-e^t)^n e^t]_{t=0}}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^k}{dt^k}[(1-e^t)^n e^t]_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \frac{d^{k+1}}{dt^{k+1}}[(1-e^t)^{n+1}]_{t=0}. \end{aligned} \tag{52}$$

But $\frac{d^{k+1}}{dt^{k+1}}[(1-e^t)^{n+1}]_{t=0} = 0$ if $n+1 \geq k+2$. Therefore, the infinite sum in (52) reduces to a finite sum

$$\begin{aligned} I^k[\eta(-t)]_{t=0} &= \sum_{n=0}^{k} \frac{1}{(n+1)^2} \frac{d^{k+1}}{dt^{k+1}}[(1-e^t)^{n+1}]_{t=0} \\ &= \sum_{n=0}^{k} \frac{1}{(n+1)^2} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \frac{d^{k+1}}{dt^{k+1}}[e^{jt}]_{t=0} \\ &= (-1)^{k+1} \sum_{n=0}^{k} \frac{1}{(n+1)^2} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} j^{k+1} \\ &= (-1)^{k+1} \sum_{n=0}^{k} \frac{1}{(n+1)^2} \Delta_{n+1}(k+1). \end{aligned} \tag{53}$$

With an appropriate change of variable in the summation index, the last equation combined with (51) gives the explicit formula for B_k:

$$B_k = (-1)^{k+1} \sum_{n=1}^{k+1} \frac{1}{n^2} \Delta_n(k+1), k \geq 0. \tag{54}$$

The extension of the last explicit formula to Bernoulli polynomials and Bernoulli numbers of fractional index is straightforward.

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